

December 1996

Degeneration of the Elliptic Algebra $\mathcal{A}_{q,p}(\widehat{sl_2})$ and Form Factors in the sine-Gordon Theory

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ABSTRACT

Following the work with Jimbo and Miwa[1], we introduce a certain degeneration of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl_2})$ and its boson realization. We investigate its rational limit. The limit is the central extension of the Yangian double $\mathcal{DY}(sl_2)$ at level one. We give a new boson realization of it. Based on these algebras, we reformulate the Smirnov's form factor bootstrap approach to the sine-Gordon theory and the $SU(2)$ invariant Thirring model. A conjectural integral formula for form factor in the sine-Gordon theory is derived.

Talk given at the Nankai-CRM joint meeting on "Extended and Quantum Algebras and their Applications to Physics", Tianjin, China, August 19-24, 1996. To appear in the CRM series in mathematical physics, Springer Verlag.

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1 Introduction

Recently, a representation theory of the quantum affine algebra $U_q(\widehat{sl_2})$ has been studied extensively and applied successfully to the XXZ model in the anti-ferromagnetic regime[2]. There the R -matrix $(R_{\varepsilon_1\varepsilon_2}^{\varepsilon'_1\varepsilon'_2}(\zeta))$, $\varepsilon_j, \varepsilon'_j = \pm j = 1, 2$ associated with the XXZ model is identified with the intertwiner of the tensor product space of the two-dimensional evaluation modules V_ζ , and the two level one infinite-dimensional highest weight modules $\mathcal{H}^{(i)}$, $i = 0, 1$ yield the exact construction of the doubly degenerated physical space of states in the thermodynamic limit. Moreover, there exist two types of intertwining operators, called type I and type II vertex operator, of the form

$$\begin{aligned} \text{Type I} \quad & \Phi^{(1-i,i)}(\zeta) : \mathcal{H}^{(i)} \longrightarrow \mathcal{H}^{(1-i)} \otimes V_\zeta, \\ & \Phi^{(1-i,i)}(\zeta) = \sum \Phi_\varepsilon^{(1-i,i)}(\zeta) \otimes v_\varepsilon, \\ \text{Type II} \quad & \Psi^{*(1-i,i)}(\zeta) : V_\zeta \otimes \mathcal{H}^{(i)} \longrightarrow \mathcal{H}^{(1-i)}, \\ & \Psi_\varepsilon^{*(1-i,i)}(\zeta) = \Psi^{*(1-i,i)}(\zeta) (v_\varepsilon \otimes \cdot). \end{aligned}$$

Each type of vertex operator plays very different role. The type I vertex operators in their certain combination give the embedding of spin operators sitting on the lattice into the physical space of states, whereas the type II vertex operator creates one physical excited particle. Hence the use of the two types of vertex operators enables us to calculate the spin-spin correlation functions as well as the form factors of the spin operators. Remarkably, the whole properties of these vertices are summarized in the following simple relations.

1. Commutation relations

$$\Phi_{\varepsilon_2}(\zeta_2)\Phi_{\varepsilon_1}(\zeta_1) = \sum_{\varepsilon'_1, \varepsilon'_2 = \pm} R_{\varepsilon_1\varepsilon_2}^{\varepsilon'_1\varepsilon'_2}(\zeta_1/\zeta_2)\Phi_{\varepsilon'_1}(\zeta_1)\Phi_{\varepsilon'_2}(\zeta_2), \quad (1.1)$$

$$\Phi_{\varepsilon_1}(\zeta_1)\Psi_{\varepsilon_2}^*(\zeta_2) = \mathcal{T}(\zeta_1/\zeta_2)\Psi_{\varepsilon_2}^*(\zeta_2)\Phi_{\varepsilon_1}(\zeta_1), \quad (1.2)$$

$$\Psi_{\varepsilon_1}^*(\zeta_1)\Psi_{\varepsilon_2}^*(\zeta_2) = \sum_{\varepsilon'_1, \varepsilon'_2 = \pm} S_{\varepsilon_1\varepsilon_2}^{\varepsilon'_1\varepsilon'_2}(\zeta_1/\zeta_2)\Psi_{\varepsilon'_2}^*(\zeta_2)\Psi_{\varepsilon'_1}^*(\zeta_1), \quad (1.3)$$

where $(S_{\varepsilon_1\varepsilon_2}^{\varepsilon'_1\varepsilon'_2}(\zeta))$ is the two-body S -matrix of the physical excited particles and $\mathcal{T}(\zeta)$ is a certain scalar function.

2. Normalization conditions

$$\Phi_{\varepsilon_1}^{(i,1-i)}(\zeta_1)\Phi_{\varepsilon_2}^{(1-i,i)}(\zeta_2) = (-1)^{1-i}\varepsilon_2g^{-1}\delta_{\varepsilon_1+\varepsilon_2,0} + O(\zeta_1 - q\zeta_2) \quad (\zeta_1 \rightarrow q\zeta_2),$$

(1.4)

$$\Psi_{\varepsilon_1}^{*(i,1-i)}(\zeta_1)\Psi_{\varepsilon_2}^{*(1-i,i)}(\zeta_2) = \frac{(-1)^{1-i}\varepsilon_1}{1-q^{-2}\zeta_2^2/\zeta_1^2} \left(\frac{\zeta_2}{q\zeta_1}\right)^{i+(1+\varepsilon_1)/2} g\delta_{\varepsilon_1+\varepsilon_2,0} + O(1)$$

$$(\zeta_1 \rightarrow q^{-1}\zeta_2), \quad (1.5)$$

with some constant g .

On the contrary to the solvable lattice models, algebraic study of the massive integrable theories has not yet been so much developed. For the on-shell S -matrix, we know that the Zamolodchikov's bootstrap approach[3], a scheme of calculation of the S -matrix, was reformulated as an algebraic problem in the representation theory of quantum groups[4, 5]. This was done in the analogous way to the R -matrix in the lattice models. On the other hand, for the off-shell quantities such as form factors of some local operators, we know Smirnov's bootstrap approach as a well-founded scheme of calculation[6]. Let $f(\beta_1, \dots, \beta_n)_{\varepsilon_1, \dots, \varepsilon_n}$ be a form factor of some local operator with rapidities β_j and spins $\varepsilon_j, j = 1, 2, \dots, n$. His approach is based on the following three axioms.

Axiom 1 The S -matrix symmetry

$$f(\beta_1, \dots, \beta_i\beta_{i+1}, \dots, \beta_n)_{\varepsilon_1, \dots, \varepsilon_i\varepsilon_{i+1}, \dots, \varepsilon_n} S_{\varepsilon_i\varepsilon_{i+1}}^{\varepsilon'_i\varepsilon'_{i+1}}(\beta_i - \beta_{i+1})$$

$$= f(\beta_1, \dots, \beta_{i+1}\beta_i, \dots, \beta_n)_{\varepsilon_1, \dots, \varepsilon'_{i+1}\varepsilon'_i, \dots, \varepsilon_n}. \quad (1.6)$$

Axiom 2

$$f(\beta_1, \dots, \beta_n + 2\pi i)_{\varepsilon_1, \dots, \varepsilon_n} = f(\beta_n, \beta_1, \dots, \beta_{n-1})_{\varepsilon_n, \varepsilon_1, \dots, \varepsilon_{n-1}}. \quad (1.7)$$

Axiom 3 As a function of β_n , form factor $f(\beta_1, \dots, \beta_n)_{\varepsilon_1, \dots, \varepsilon_n}$ is an analytic function in the strip $0 \leq \text{Im}\beta_n \leq 2\pi$ and has only simple poles at $\beta_n = \beta_j + \pi i, \quad n > j$. The corresponding residues are given by

$$2\pi i \text{ res } f(\beta_1, \dots, \beta_n)_{\varepsilon_1, \dots, \varepsilon_n}$$

$$= f(\beta_1, \dots, \hat{\beta}_j, \dots, \beta_{n-1})_{\varepsilon'_1, \dots, \varepsilon'_j, \dots, \varepsilon'_{n-1}} \mathcal{C}_{\varepsilon_n, \varepsilon'_j}$$

$$\times \left(\delta_{\varepsilon_1}^{\varepsilon'_1} \cdots \delta_{\varepsilon_{j-1}}^{\varepsilon'_{j-1}} S_{\varepsilon_{n-1}\tau_1}^{\varepsilon'_{n-1}\varepsilon'_j}(\beta_{n-1} - \beta_j) S_{\varepsilon_{n-2}\tau_2}^{\varepsilon'_{n-2}\tau_1}(\beta_{n-2} - \beta_j) \cdots S_{\varepsilon_{j+1}\varepsilon_j}^{\varepsilon'_{j+1}\tau_{n-j-2}}(\beta_{j+1} - \beta_j) \right.$$

$$\left. - S_{\tau_1\varepsilon_1}^{\varepsilon'_j\varepsilon'_1}(\beta_j - \beta_1) \cdots S_{\tau_{j-2}\varepsilon_{j-2}}^{\tau_{j-3}\varepsilon'_{j-2}}(\beta_j - \beta_{j-2}) S_{\varepsilon_j\varepsilon_{j-1}}^{\tau_{j-2}\varepsilon'_{j-1}}(\beta_j - \beta_{j-1}) \delta_{\varepsilon_{j+1}}^{\varepsilon'_{j+1}} \cdots \delta_{\varepsilon_{n-1}}^{\varepsilon'_{n-1}} \right), \quad (1.8)$$

where \mathcal{C} is the charge conjugation matrix.

These axioms define a matrix Riemann-Hilbert problem for the functions $f(\beta_1, \dots, \beta_n)_{\epsilon_1, \dots, \epsilon_n}$. Remarkably, quite similar properties to (1.6)-(1.8) are satisfied by the correlation functions and form factors in the lattice models (See for example [15]). Hence one may expect that *there must be some algebraic formulation of the Smirnov's form factor bootstrap approach*.

In fact, in [7], Smirnov considered the $SU(2)$ invariant Thirring model, which is a certain limit of the sine-Gordon theory, and proposed that the Yangian double $\mathcal{DY}(sl_2)$ is a relevant algebra. He conjectured also that the level-0 representation is a relevant representation[7]. However due to lack of infinite dimensional representation, he failed to reproduce his whole axioms.

The second progress was brought by Lukyanov[8]. He considered the sine-Gordon theory and investigated an algebra of vertex operators (3.15)-(3.17) which are very similar to (1.1)-(1.5). He showed that such algebra formally reproduces Smirnov's whole axioms. However, his argument is rather a phenomenological one. No one had succeeded in giving it any representation theoretical foundation until Ref.[1].

In this paper, according to the prior work [1], we clarify an underlying quantum group structure of Smirnov's form factor bootstrap and Lukyanov's algebra. We introduce a certain degeneration of the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl_2})$ [9] and present a boson representation of it. We investigate also its rational limit and identify it with the central extension of the Yangian double $\mathcal{DY}(sl_2)$ at level one. We show that in the both cases certain gauge transformations of the type I and the type II vertex operators satisfy the Lukyanov's algebra. As a result, we give formulae for form factors, in the case of the sine-Gordon theory and the $SU(2)$ invariant Thirring model, which satisfy the whole Smirnov's axioms, based on the representations of our algebras. This is natural because the sine-Gordon theory is known as a continuum limit of the XYZ model[10], and the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl_2})$ is the algebra which is conjectured to give an algebraic foundation of the XYZ model (See for example [11]).

We hence obtain a unified way, whose key formulae are given by (1.1)-(1.5), to formulate correlation functions and form factors both in the exactly solvable lattice models and the massive integrable quantum field theories. We finally give a conjectural integral formula for form factor in the sine-Gordon theory.

2 Elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl_2})$

Let us begin with the Baxter's elliptic R matrix[12].

$$R(\zeta) = R(\zeta; p^{1/2}, q^{1/2}) = \frac{1}{\mu(\zeta)} \begin{pmatrix} a(u) & & & d(u) \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ d(u) & & & a(u) \end{pmatrix}, \quad (2.1)$$

$$a(u) = \frac{\sinh(\lambda - u)}{\sinh(\lambda)}, \quad b(u) = \frac{\sinh(u)}{\sinh(\lambda)}, \quad c(u) = 1, \quad d(u) = k \sinh(\lambda - u) \sinh(u), \quad (2.2)$$

where $\sinh(u) = -i \operatorname{sn}(iu)$, and $\operatorname{sn}(u)$ is Jacobi's elliptic function with modulus k . Let K, K' be the corresponding complete elliptic integrals. We use also the variables

$$p = e^{-\frac{\pi K'}{K}}, \quad q = -e^{-\frac{\pi \lambda}{2K}}, \quad \zeta = e^{\frac{\pi u}{2K}}, \quad (2.3)$$

and regard (2.2) as functions of ζ, p, q . We choose the overall scalar factor $\mu(\zeta)$ as follows.

$$\begin{aligned} \frac{1}{\mu(\zeta)} &= \frac{1}{\bar{\kappa}(\zeta^2)} \frac{(p^2; p^2)_\infty}{(p; p)_\infty^2} \frac{\Theta_{p^2}(q^2) \Theta_{p^2}(p\zeta^2)}{\Theta_{p^2}(q^2 \zeta^2)}, \\ \frac{1}{\bar{\kappa}(z)} &= \frac{(q^4 z^{-1}; p, q^4)_\infty (q^2 z; p, q^4)_\infty (pz^{-1}; p, q^4)_\infty (pq^2 z; p, q^4)_\infty}{(q^4 z; p, q^4)_\infty (q^2 z^{-1}; p, q^4)_\infty (pz; p, q^4)_\infty (pq^2 z^{-1}; p, q^4)_\infty}, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} (z; p_1, \dots, p_m)_\infty &= \prod_{n_1, \dots, n_m \geq 0} (1 - zp_1^{n_1} \dots p_m^{n_m}), \\ \Theta_q(z) &= (z; q)_\infty (qz^{-1}; q)_\infty (q; q)_\infty. \end{aligned}$$

Let us consider the formal generating series

$$\begin{aligned} L^\pm(\zeta) &= \sum_{n=-\infty}^{\infty} L_n^\pm \zeta^{-n}, \quad L_n^\pm = \left(L_{\varepsilon \varepsilon', n}^\pm \right)_{\varepsilon, \varepsilon' = \pm}, \\ L_{\varepsilon \varepsilon', n}^\pm &= 0 \quad \text{if } \varepsilon \varepsilon' \neq (-1)^n. \end{aligned} \quad (2.5)$$

Definition : the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl_2})$ [9, 13]

The elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl_2})$ is the algebra generated by the symbols $L_{\varepsilon\varepsilon',n}^{\pm}$ ($n \in \mathbf{Z}, \varepsilon, \varepsilon' = \pm, \varepsilon\varepsilon' = (-1)^n$) and a central element c , through the following relations.

$$R_{12}^{\pm}(\zeta_1/\zeta_2) \overset{1}{L}^{\pm}(\zeta_1) \overset{2}{L}^{\pm}(\zeta_2) = \overset{2}{L}^{\pm}(\zeta_2) \overset{1}{L}^{\pm}(\zeta_1) R_{12}^{*\pm}(\zeta_1/\zeta_2), \quad (2.6)$$

$$R_{12}^{+}(q^{c/2}\zeta_1/\zeta_2) \overset{1}{L}^{+}(\zeta_1) \overset{2}{L}^{-}(\zeta_2) = \overset{2}{L}^{-}(\zeta_2) \overset{1}{L}^{+}(\zeta_1) R_{12}^{*+}(q^{-c/2}\zeta_1/\zeta_2), \quad (2.7)$$

$$q\text{-det}L^{+}(\zeta) \equiv L_{++}^{+}(q^{-1}\zeta)L_{--}^{+}(\zeta) - L_{-+}^{+}(q^{-1}\zeta)L_{+-}^{+}(\zeta) = q^{c/2}, \quad (2.8)$$

$$L_{\varepsilon\varepsilon'}^{-}(\zeta) = \varepsilon\varepsilon' L_{-\varepsilon,-\varepsilon'}^{+}(p^{1/2}q^{-c/2}\zeta), \quad (2.9)$$

where

$$R^{+}(\zeta) = \tau(q^{1/2}\zeta^{-1})R(\zeta), \quad R^{-}(\zeta) = \tau(q^{1/2}\zeta)^{-1}R(\zeta)$$

with

$$\tau(\zeta) = \zeta^{-1} \frac{(q\zeta^2; q^4)_{\infty} (q^3\zeta^{-2}; q^4)_{\infty}}{(q^3\zeta^2; q^4)_{\infty} (q\zeta^{-2}; q^4)_{\infty}} \quad (2.10)$$

and

$$R^{*\pm}(\zeta) = R^{\pm}(\zeta; p^{*1/2}, q^{1/2}), \quad p^{*} = pq^{-2c}. \quad (2.11)$$

In [9], it was conjectured that the elliptic algebra $\mathcal{A}_{q,p}(\widehat{sl_2})$ has natural analogs of the level one modules $\mathcal{H}^{(i)}$ $i=0,1$ ¹ and vertex operators $\Phi_{\varepsilon}^{(1-i,i)}(\zeta)$ and $\Psi_{\varepsilon}^{*(1-i,i)}(\zeta)$. It was also conjectured that these vertex operators satisfy the commutation relation (1.1)-(1.5) with the elliptic R -matrix (2.1), $S_{\varepsilon_1, \varepsilon_2}^{\varepsilon'_1, \varepsilon'_2}(\zeta) = -R_{\varepsilon_1, \varepsilon_2}^{*\varepsilon'_1, \varepsilon'_2}(\zeta)$, $\mathcal{T}(\zeta) = \tau(\zeta)$ and a different constant g .

In terms of the vertex operators, the L^{\pm} operators acting on $\mathcal{H}^{(i)}$ can be expressed as follows.

$$L_{\varepsilon\varepsilon'}^{+}(\zeta) = \kappa \Psi_{\varepsilon'}^{*}(\zeta) \Phi_{\varepsilon}(q^{1/2}\zeta), \quad (2.12)$$

$$L_{\varepsilon\varepsilon'}^{-}(\zeta) = \kappa \Phi_{\varepsilon}(\zeta) \Psi_{\varepsilon'}^{*}(q^{1/2}\zeta). \quad (2.13)$$

Here κ is a normalization constant. Then the defining relations (2.6),(2.7) are immediate consequences of the elliptic analogue of the commutation relations (1.1) -(1.3). The condition (2.8) for the quantum determinant also

¹ We say that a representation of $\mathcal{A}_{q,p}(\widehat{sl_2})$ has level k if the central element c acts as k times the identity.

follows from (1.4),(1.5) with an appropriate choice of κ . The symmetry (2.9) of the L operators entails the following relation for the vertex operators.

$$\Phi_\varepsilon(\zeta)\Psi_{\varepsilon'}^*(q^{1/2}\zeta) = \varepsilon\varepsilon'\Psi_{-\varepsilon'}^*(p^{1/2}q^{-1/2}\zeta)\Phi_{-\varepsilon}(p^{1/2}\zeta). \quad (2.14)$$

3 Degeneration of $\mathcal{A}_{q,p}(\widehat{sl_2})$

3.1 Trigonometric limit

There are two interesting degeneration limit.

- 1) $K \rightarrow \frac{\pi}{2}, K' \rightarrow \infty$ i.e. $p \rightarrow 0, q \rightarrow -e^{-\lambda}$.

Let us set

$$L_{\varepsilon\varepsilon',n}^+ = \left(-p^{1/2}\right)^{\max(n,0)} \overline{L}_{\varepsilon\varepsilon',n}^+$$

and let formally $p \rightarrow 0$, then $L^+(\zeta)$ and $L^-(\zeta)$ become power series in ζ and ζ^{-1} respectively. In this limit, the relations (2.6)-(2.9) reduce to the defining relations of the quantum affine algebra $U_q(\widehat{sl_2})$ due to Reshetikhin and Semenov-Tian-Shanskii [14].

- 2) $K \rightarrow \infty, K' \rightarrow \frac{\pi}{2}$. More precisely, we let[1]

$$p = q^{2(\xi+1)}, \quad \zeta = q^{i\beta/\pi}, \quad q \rightarrow 1 \quad (3.1)$$

with ξ and β being kept fixed.

In this limit, the elliptic R matrices (2.1) and $R^*(\zeta) = R(\zeta; p^{*1/2}; q^{1/2})$ degenerate to trigonometric ones.

$$\begin{aligned} \tilde{R}^*(\beta) &= \lim R(\zeta; p^{*1/2}; q^{1/2}) \\ &= -S_0(\beta) \begin{pmatrix} \frac{\cosh \frac{i\pi}{2\xi} \cosh \frac{\beta}{2\xi}}{\cosh \frac{i\pi-\beta}{2\xi}} & & & -\frac{\sinh \frac{i\pi}{2\xi} \sinh \frac{\beta}{2\xi}}{\cosh \frac{i\pi-\beta}{2\xi}} \\ & -\frac{\cosh \frac{i\pi}{2\xi} \sinh \frac{\beta}{2\xi}}{\sinh \frac{i\pi-\beta}{2\xi}} & \frac{\sinh \frac{i\pi}{2\xi} \cosh \frac{\beta}{2\xi}}{\sinh \frac{i\pi-\beta}{2\xi}} & \\ & \frac{\sinh \frac{i\pi}{2\xi} \cosh \frac{\beta}{2\xi}}{\sinh \frac{i\pi-\beta}{2\xi}} & -\frac{\cosh \frac{i\pi}{2\xi} \sinh \frac{\beta}{2\xi}}{\sinh \frac{i\pi-\beta}{2\xi}} & \\ -\frac{\sinh \frac{i\pi}{2\xi} \sinh \frac{\beta}{2\xi}}{\cosh \frac{i\pi-\beta}{2\xi}} & & & \frac{\cosh \frac{i\pi}{2\xi} \cosh \frac{\beta}{2\xi}}{\cosh \frac{i\pi-\beta}{2\xi}} \end{pmatrix} \end{aligned} \quad (3.2)$$

$$\begin{aligned} \tilde{R}(\beta) &= \lim R(\zeta; p^{1/2}; q^{1/2}) \\ &= -\tilde{R}^*(-\beta)|_{\xi \rightarrow \xi+1}. \end{aligned} \quad (3.3)$$

Here $S_0(\beta)$ is given by

$$S_0(\beta) = \frac{S_2(-i\beta)S_2(\pi + i\beta)}{S_2(i\beta)S_2(\pi - i\beta)} \quad (3.4)$$

with $S_2(x)$ being Barnes' double sine function with periods 2π and $\pi\xi$ [15]. These are not the standard trigonometric R matrix coming from the universal R matrix of $U_q(\widehat{sl}_2)$. In order to bring them to the usual form, we need to introduce a 'gauge' transformation[11]. Define

$$U = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \quad U_0 = U \sigma^z, \quad U_1 = \sigma^z U \sigma^z.$$

Then we find

$$(U_1 \otimes U_0) \tilde{R}(\beta) (U_0 \otimes U_1)^{-1} = R(-\beta), \quad (3.5)$$

$$(U_1 \otimes U_0) \left(-\tilde{R}^*(\beta) \right) (U_0 \otimes U_1)^{-1} = S(\beta) \quad (3.6)$$

with

$$S(\beta) = \frac{S_0(\beta)}{\sinh \frac{i\pi - \beta}{\xi}} \begin{pmatrix} \sinh \frac{i\pi - \beta}{\xi} & & & \\ & \sinh \frac{\beta}{\xi} & \sinh \frac{i\pi}{\xi} & \\ & \sinh \frac{i\pi}{\xi} & \sinh \frac{\beta}{\xi} & \\ & & & \sinh \frac{i\pi - \beta}{\xi} \end{pmatrix}, \quad (3.7)$$

$$R(\beta) = -S(-\beta)|_{\xi \rightarrow \xi+1}. \quad (3.8)$$

We have also

$$(U_0 \otimes U_1) \tilde{R}(\beta) (U_1 \otimes U_0)^{-1} = R(-\beta), \quad (3.9)$$

$$(U_0 \otimes U_1) \left(-\tilde{R}^*(\beta) \right) (U_1 \otimes U_0)^{-1} = S(\beta). \quad (3.10)$$

The matrices $(S_{ab}^{cd}(\beta))$, $(R_{ab}^{cd}(\beta))$ coincide with the two-body S -matrix of the sine-Gordon theory[3] and the R -matrix of the XXZ model in the gapless regime[15].

Now we are interested in the degeneration limit of the elliptic analogue of the relations (1.1)-(1.3). Write $\Phi_a(\beta)$, $\Psi_a^*(\beta)$ for the limit of $\Phi_a(\zeta)$, $\Psi_a^*(\zeta)$. We set

$$Z_a^{(1,0)}(\beta) = 2 \sum_b (U_0^{-1})_{ba} \Psi_b^{*(1,0)}(\beta) = \Psi_+^{*(1,0)}(\beta) - ia \Psi_-^{*(1,0)}(\beta) \quad (3.11)$$

$$Z_a^{(0,1)}(\beta) = 2 \sum_b (U_1^{-1})_{ba} \Psi_b^{*(0,1)}(\beta) = a \Psi_+^{*(0,1)}(\beta) - i \Psi_-^{*(0,1)}(\beta) \quad (3.12)$$

$$Z'_a{}^{(1,0)}(\beta) = \sum_b (U_0)_{ab} \Phi_b^{(1,0)}(\beta) = \Phi_+^{(1,0)}(\beta) + i a \Phi_-^{(1,0)}(\beta), \quad (3.13)$$

$$Z'_a{}^{(0,1)}(\beta) = \sum_b (U_1)_{ab} \Phi_b^{(0,1)}(\beta) = a \Phi_+^{(0,1)}(\beta) + i \Phi_-^{(0,1)}(\beta). \quad (3.14)$$

Their commutation relations can be determined using (3.5) and (3.6). Dropping the upper indices, we find

$$Z_a(\beta_1) Z_b(\beta_2) = \sum_{c,d} S_{ab}^{cd}(\beta_1 - \beta_2) Z_d(\beta_2) Z_c(\beta_1), \quad (3.15)$$

$$Z'_a(\beta_1) Z'_b(\beta_2) = \sum_{c,d} R_{ab}^{cd}(\beta_1 - \beta_2) Z'_d(\beta_2) Z'_c(\beta_1), \quad (3.16)$$

$$Z_a(\beta_1) Z'_b(\beta_2) = ab \tan\left(\frac{\pi}{4} + i \frac{\beta_1 - \beta_2}{2}\right) Z'_b(\beta_2) Z_a(\beta_1). \quad (3.17)$$

Here we have used

$$\lim \tau(q^{i\beta/\pi}) = \tan\left(\frac{\pi}{4} + \frac{i\beta}{2}\right). \quad (3.18)$$

The conditions corresponding to (1.4) and (1.5) become

$$Z_a(\beta_1) Z_b(\beta_2) = \frac{C}{\beta_1 - \beta_2 - \pi i} \delta_{a+b,0} + O(1) \quad (\beta_1 \rightarrow \beta_2 + \pi i), \quad (3.19)$$

$$Z'_a(\beta) Z'_b(\beta + \pi i) = C' \delta_{a+b,0}. \quad (3.20)$$

Here C, C' are constants depending on the normalization of $Z_a(\beta), Z'_a(\beta)$. In addition, the symmetry relation (2.14) reduces to the following.

$$Z'_a(\beta) Z_b(\beta - \frac{\pi i}{2}) = Z_b(\beta^*) Z'_a(\beta^* - \frac{\pi i}{2}), \quad \beta^* = \beta - \pi i(\xi + \frac{1}{2}). \quad (3.21)$$

One should note that the relations (3.15)-(3.17) are much simpler than those for $\Phi_\varepsilon(\beta)$ and $\Psi_\varepsilon^*(\beta)$. In the next section we will give a boson representation of $Z_a(\beta)$ and $Z'_a(\beta)$.

Remark. Let us denote the limit of the L -operators $L_{\varepsilon, \varepsilon'}^\pm(\zeta)$ by $L_{\varepsilon, \varepsilon'}^\pm(\beta)$. Then the defining relation of the elliptic algebra (2.6)-(2.9) degenerate to

$$\tilde{R}_{12}^\pm(\beta_1 - \beta_2) L_1^{\pm 1}(\beta_1) L_2^{\pm 2}(\beta_2) = L_2^{\pm 2}(\beta_2) L_1^{\pm 1}(\beta_1) \tilde{R}_{12}^{*\pm}(\beta_1 - \beta_2), \quad (3.22)$$

$$\tilde{R}_{12}^+(\beta_1 - \beta_2 - \frac{i\pi}{2}) L^+{}^1(\beta_1) L^-{}^2(\beta_2) = L^-{}^2(\beta_2) L^+{}^1(\beta_1) \tilde{R}_{12}^{*+}(\beta_1 - \beta_2 + \frac{i\pi}{2}), \quad (3.23)$$

$$q\text{-det} L^+(\beta) \equiv L_{++}^+(\beta + i\pi) L_{--}^+(\beta) - L_{-+}^+(\beta + i\pi) L_{+-}^+(\beta) = 1, \quad (3.24)$$

$$L_{\varepsilon\varepsilon'}^-(\beta) = \varepsilon\varepsilon' L_{-\varepsilon, -\varepsilon'}^+(\beta - i\pi(\xi + \frac{1}{2})). \quad (3.25)$$

These are similar to the defining relations of the quantum affine algebra $U_q(\widehat{sl_2})$ [14]. However the extra relation (3.25) indicates that our L -operators $L^\pm(\beta)$ have no Gauss decomposition. Our resultant algebra is hence quite different from $U_q(\widehat{sl_2})$ with $|q| = 1$.

3.2 Rational limit

We next consider the rational limit $\xi \rightarrow \infty$ of the results in §3.1. In this limit, we have

$$\lim_{\xi \rightarrow \infty} \tilde{R}^*(\beta) = S_R(\beta), \quad \lim_{\xi \rightarrow \infty} \tilde{R}(\beta) = R_R(\beta) \quad (3.26)$$

with

$$S_R(\beta) = S_{R,0}(\beta) \begin{pmatrix} 1 & & \\ & \frac{\beta}{i\pi - \beta} & \frac{i\pi}{i\pi - \beta} \\ & \frac{i\pi}{i\pi - \beta} & \frac{\beta}{i\pi - \beta} \\ & & & 1 \end{pmatrix}, \quad (3.27)$$

$$S_{R,0}(\beta) = \frac{\Gamma(\frac{i\beta}{2\pi})\Gamma(\frac{1}{2} - \frac{i\beta}{2\pi})}{\Gamma(-\frac{i\beta}{2\pi})\Gamma(\frac{1}{2} + \frac{i\beta}{2\pi})} \quad (3.28)$$

and

$$R_R(\beta) = -S_R(-\beta). \quad (3.29)$$

Note that $S_R(\beta)$ and $R_R(\beta)$ are invariant under the transformations (3.5)-(3.6) and (3.9)-(3.10). They coincide with the S -matrix of the $SU(2)$ invariant Thirring model and the R -matrix of the XXX model, respectively.

Let $L_{R\varepsilon, \varepsilon'}^\pm(\beta)$ be the rational limit of $L_{\varepsilon, \varepsilon'}^\pm(\beta)$. In this limit, the relations (3.22)-(3.24) are still hold with replacement $\tilde{R}^*(\beta) \rightarrow S_R(\beta)$, $\tilde{R}(\beta) \rightarrow R_R(\beta)$. However the last relation (3.25) is broken down since the RHS loses its meaning. Hence the resultant algebra is the central extension of the Yangian double $\mathcal{DY}(sl_2)$ at level one due to Reshetikhin and Semenov-Tian-Shanskii [14].

4 Bosonization of the vertex operators

We here consider the boson representation of the algebra (3.15)-(3.20).

4.1 Trigonometric case

Let us consider free bosons $a(t)$ ($t \in \mathbf{R}$) which satisfy[1]

$$[a(t), a(t')] = \frac{\sinh \frac{\pi t}{2} \sinh \pi t \sinh \frac{\pi t(\xi+1)}{2}}{t \sinh \frac{\pi t \xi}{2}} \delta(t+t'). \quad (4.1)$$

We also use $a'(t)$ defined by

$$a'(t) \sinh \frac{\pi t(\xi+1)}{2} = a(t) \sinh \frac{\pi t \xi}{2}.$$

We consider the Fock space \mathcal{H} generated by $|\text{vac}\rangle$ which satisfies

$$a(t)|\text{vac}\rangle = 0 \quad \text{if } t > 0.$$

We set

$$\begin{aligned} V(\alpha) &= : e^{i\phi(\alpha)} :, & i\phi(\alpha) &= \int_{-\infty}^{\infty} \frac{a(t)}{\sinh \pi t} e^{i\alpha t} dt, \\ \bar{V}(\alpha) &= : e^{-i\bar{\phi}(\alpha)} :, & i\bar{\phi}(\alpha) &= \int_{-\infty}^{\infty} \frac{a(t)}{\sinh \frac{\pi t}{2}} e^{i\alpha t} dt, \\ V'(\alpha) &= : e^{i\phi'(\alpha)} :, & i\phi'(\alpha) &= - \int_{-\infty}^{\infty} \frac{a'(t)}{\sinh \pi t} e^{i\alpha t} dt, \\ \bar{V}'(\alpha) &= : e^{-i\bar{\phi}'(\alpha)} :, & i\bar{\phi}'(\alpha) &= - \int_{-\infty}^{\infty} \frac{a'(t)}{\sinh \frac{\pi t}{2}} e^{i\alpha t} dt. \end{aligned}$$

In Appendix 1 in Ref.[1], one can find the list of the operator products of these operators.

Let us define

$$Z_+(\beta) = V(\beta), \quad (4.2)$$

$$Z_-(\beta) = \int_{C_1} \frac{d\alpha}{2\pi} : e^{i\phi(\beta) - i\bar{\phi}(\alpha)} : f(\alpha - \beta), \quad (4.3)$$

$$Z'_+(\beta) = V'(\beta), \quad (4.4)$$

$$Z'_-(\beta) = \int_{C_2} \frac{d\alpha}{2\pi} : e^{i\phi'(\beta) - i\bar{\phi}'(\alpha)} : f'(\alpha - \beta), \quad (4.5)$$

where²

$$\begin{aligned}
f(\alpha) &= c_1 \sinh \frac{\pi}{\xi} \Gamma\left(\frac{i\alpha}{\pi\xi} - \frac{1}{2\xi}\right) \Gamma\left(-\frac{i\alpha}{\pi\xi} - \frac{1}{2\xi}\right), & c_1 &= \frac{e^{-(\gamma+\log \pi\xi)\frac{\xi}{\xi+1}}}{i\pi}, \\
f'(\alpha) &= c_2 \sinh \frac{\pi}{\xi+1} \Gamma\left(\frac{i\alpha}{\pi(\xi+1)} + \frac{1}{2(\xi+1)}\right) \Gamma\left(-\frac{i\alpha}{\pi(\xi+1)} + \frac{1}{2(\xi+1)}\right), \\
c_2 &= \frac{e^{-(\gamma+\log \pi(\xi+1))\frac{\xi+1}{\xi}}}{i\pi}.
\end{aligned}$$

Here the integration contours are chosen as follows. The contour C_1 is $(-\infty, \infty)$ except that the poles $\beta - \frac{\pi i}{2} + n\pi\xi i$ ($n \in \mathbf{Z}_{\geq 0}$) of $\Gamma(\frac{i(\alpha-\beta)}{\pi\xi} - \frac{1}{2\xi})$ are above C_1 and the poles $\beta + \frac{\pi i}{2} - n\pi\xi i$ ($n \in \mathbf{Z}_{\geq 0}$) of $\Gamma(-\frac{i(\alpha-\beta)}{\pi\xi} - \frac{1}{2\xi})$ are below C_1 . The contour C_2 is $(-\infty, \infty)$. The poles $\beta + \frac{\pi i}{2} + n\pi(\xi+1)i$ ($n \in \mathbf{Z}_{\geq 0}$) of $\Gamma(\frac{i(\alpha-\beta)}{\pi(\xi+1)} + \frac{1}{2(\xi+1)})$ are above C_2 and the poles $\beta - \frac{\pi i}{2} - n\pi(\xi+1)i$ ($n \in \mathbf{Z}_{\geq 0}$) of $\Gamma(-\frac{i(\alpha-\beta)}{\pi(\xi+1)} + \frac{1}{2(\xi+1)})$ are below C_2 .

In [1], we proved the following statement.

Proposition 4.1 *The operators $Z_{\pm}(\beta)$ and $Z'_{\pm}(\beta)$ satisfy the commutation relations (3.15)-(3.17), the normalization conditions (3.19) and (3.20) as well as the symmetry relation (3.21). The constants C, C' are given by*

$$\begin{aligned}
C &= \left(\pi\xi c_1 \sin \frac{\pi}{\xi} \Gamma\left(-\frac{1}{\xi}\right) \right)^2 g(-\pi i), \\
C' &= \left(\pi(\xi+1) c'_1 \sin \frac{\pi}{\xi+1} \Gamma\left(\frac{1}{\xi+1}\right) \right)^2 \lim_{\beta \rightarrow 0} \frac{g'(\beta + \pi i)}{\beta}.
\end{aligned}$$

4.2 Rational case

Let us next consider the rational limit. We denote the limit $\xi \rightarrow \infty$ of $a(t)$ and $a'(t)$ by $a_R(t)$ and $a'_R(t)$, respectively. They satisfy

$$[a_R(t), a_R(t')] = \frac{\sinh \frac{\pi t}{2} \sinh \pi t e^{\frac{\pi|t|}{2}}}{t} \delta(t+t'). \quad (4.6)$$

$$a'_R(t) e^{\frac{\pi|t|}{2}} = a_R(t). \quad (4.7)$$

² In order to make the expressions admit their rational limit, we have changed the definition of $f(\alpha)$ and $f'(\alpha)$ by adding extra constant factors omitted in Ref.[1].

We denote the corresponding limit of the boson fields, vertex operators and the Fock space by adding the suffix R . For example, $i\phi_R(\beta) = \lim_{\xi \rightarrow \infty} i\phi(\beta) = \int_{-\infty}^{\infty} \frac{a_R(t)}{\sinh \pi t} e^{i\alpha t} dt$. Under these notations, the rational limit of the vertex operators $Z_{\pm}(\beta)$ and $Z'_{\pm}(\beta)$ are given by

$$Z_{R+}(\beta) = V_R(\beta), \quad (4.8)$$

$$Z_{R-}(\beta) = \int_{C_{R1}} \frac{d\alpha}{2\pi} : e^{i\phi_R(\beta) - i\bar{\phi}_R(\alpha)} : f_R(\alpha - \beta), \quad (4.9)$$

$$Z'_{R+}(\beta) = V'_R(\beta), \quad (4.10)$$

$$Z'_{R-}(\beta) = \int_{C_{R2}} \frac{d\alpha}{2\pi} : e^{i\phi'_R(\beta) - i\bar{\phi}'_R(\alpha)} : f'_R(\alpha - \beta), \quad (4.11)$$

where

$$f_R(\alpha) = f'_R(\alpha) = -i\pi e^{-\gamma} \frac{1}{\alpha^2 + \frac{\pi^2}{4}}.$$

The integration contours C_{R1} , C_{R2} should be chosen as follows. The contour C_{R1} is $(-\infty, \infty)$ except that the pole $\beta - \frac{\pi i}{2}$ is above C_{R1} and the pole $\beta + \frac{\pi i}{2}$ is below C_{R1} . The contour C_{R2} is $(-\infty, \infty)$. The pole $\beta + \frac{\pi i}{2}$ is above C_{R2} and the pole $\beta - \frac{\pi i}{2}$ is below C_{R2} .

In Appendix, we list all the operator products of the vertex operators $V_R(\alpha)$, $\bar{V}_R(\alpha)$, $V'_R(\alpha)$ and $\bar{V}'_R(\alpha)$.

After changing the definition of $f(\alpha)$ and $f'(\alpha)$ to (4.6) and (4.6) in the proof of Prop.3.1 and 3.3 in [1], whole arguments given there admit their rational limits. Hence we obtain

Proposition 4.2 *The operators $Z_{R\pm}(\beta)$ and $Z'_{R\pm}(\beta)$ satisfy the rational limit of the commutation relations (3.15)-(3.17) and the normalization conditions (3.19)(3.20). The corresponding constants C_R, C'_R are given by*

$$C_R = -\sqrt{2}e^{-\frac{3\gamma}{2}},$$

$$C'_R = -i\sqrt{2}e^{-\frac{3\gamma}{2}}$$

Hence the operators $L_R^{\pm}(\beta)$

$$L_{R\varepsilon\varepsilon'}^+(\beta) = \kappa_R \Psi_{R\varepsilon'}^*(\beta) \Phi_{R\varepsilon}(\beta - \frac{\pi i}{2}), \quad (4.12)$$

$$L_{R\varepsilon\varepsilon'}^-(\beta) = \kappa_R \Phi_{\varepsilon}(\beta) \Psi_{R\varepsilon'}^*(\beta - \frac{\pi i}{2}) \quad (4.13)$$

with (4.8)-(4.11) and the gauge transformation (3.5)-(3.10) gives a boson representation of the central extension of the Yangian double at level one. One should compare this with those in [16, 17].

5 Form factor in the sine-Gordon theory

Let us define the boost operator H by

$$H = \int_0^\infty dt \frac{t^2 \sinh \frac{\pi t(\xi+1)}{2}}{\sinh \frac{\pi t}{2} \sinh \pi t \sinh \frac{\pi t \xi}{2}} a'(-t) a'(t), \quad (5.1)$$

which enjoys the property

$$e^{\lambda H} a'(t) e^{-\lambda H} = e^{-\lambda t} a'(t). \quad (5.2)$$

Hence we have

$$e^{\lambda H} X(\beta) e^{-\lambda H} = X(\beta + i\lambda), \quad (5.3)$$

for $X = V, \bar{V}, V', \bar{V}'$.

Let us consider the operators

$$\mathcal{O}(\alpha_1, \dots, \alpha_m)_{\varepsilon_1, \dots, \varepsilon_m} = Z'_{-\varepsilon_1}(\alpha_1 + \pi i) \cdots Z'_{-\varepsilon_m}(\alpha_m + \pi i) Z'_{\varepsilon_m}(\alpha_m) \cdots Z'_{\varepsilon_1}(\alpha_1)$$

Using (3.17), we have

Proposition 5.1

$$[\mathcal{O}(\alpha_1, \dots, \alpha_m), Z_{\pm}(\beta)] = 0 \quad \forall \beta, \alpha_j \ (j = 1, \dots, m). \quad (5.4)$$

Now let us consider the following function.

$$\begin{aligned} & F^{\mathcal{O}}(\beta_1, \dots, \beta_N)_{\mu_1, \dots, \mu_N} \\ &= \frac{\text{tr}_{\mathcal{H}}(e^{-\lambda H} \mathcal{O}(\alpha_1, \dots, \alpha_m) Z_{\mu_N}(\beta_N) Z_{\mu_{N-1}}(\beta_{N-1}) \cdots Z_{\mu_1}(\beta_1))}{\text{tr}_{\mathcal{H}}(e^{-\lambda H})}. \end{aligned} \quad (5.5)$$

By using the relations (3.15), (5.3) and the cyclic property of trace, one can show that the function $F^{\mathcal{O}}(\beta_1, \dots, \beta_{2n})_{\mu_1, \dots, \mu_{2n}}$ satisfies the Smirnov's first axiom with the S -matrix of the sine-Gordon theory and the following level zero deformed Knizhnik-Zamolodchikov equation.

$$\begin{aligned} & F^{\mathcal{O}}(\beta_1, \dots, \beta_N + i\lambda)_{\mu_1, \dots, \mu_N} \\ &= S_{\mu_1 \mu_N}^{\mu'_1 \tau_1}(\beta_1 - \beta_N) S_{\mu_2 \tau_1}^{\mu'_2 \tau_2}(\beta_2 - \beta_N) \cdots S_{\mu_{N-1} \tau_{N-2}}^{\mu'_{N-1} \mu'_N}(\beta_{N-1} - \beta_N) \\ & \quad \times F^{\mathcal{O}}(\beta_1, \dots, \beta_N)_{\mu'_1, \dots, \mu'_N}. \end{aligned} \quad (5.6)$$

If one sets $\lambda = 2\pi$, the S -matrix symmetry and the deformed KZ equation are equivalent to the Smirnov's first and second axioms.

In order to show that the function $F^\mathcal{O}(\beta_1, \dots, \beta_N)_{\mu_1, \dots, \mu_N}$ satisfies the third axiom, let us consider the relation (3.19). Applying this to the product $Z_{\mu_N}(\beta_N)Z_{\mu_{N-1}}(\beta_{N-1})$ in (5.5), one finds a simple pole at $\beta_N = \beta_{N-1} + \pi i$. In addition, noting the cyclic property of the trace and using (3.15) and (5.3), one can change the order of $Z_{\mu_N}(\beta_N)$ and $Z_{\mu_{N-1}}(\beta_{N-1})$ as follows.

$$\begin{aligned} & S_{\mu_{N-1}\mu_{N-2}}^{\tau_1\tau'_{N-2}}(\beta_{N-1} - \beta_{N-2}) S_{\tau_1\mu_{N-3}}^{\tau_2\mu'_{N-3}}(\beta_{N-1} - \beta_{N-3}) \cdots S_{\tau_{N-2}\mu_1}^{\mu'_{N-1}\mu'_1}(\beta_{N-1} - \beta_1) \\ & \times \frac{\text{tr}_{\mathcal{H}}(e^{-\lambda H} \mathcal{O}(\alpha_1, \dots, \alpha_m) Z_{\mu_{N-1}}(\beta_{N-1} + i\lambda) Z_{\mu_N}(\beta_N) Z_{\mu_{N-2}}(\beta_{N-2}) \cdots Z_{\mu_1}(\beta_1))}{\text{tr}_{\mathcal{H}}(e^{-\lambda H})}. \end{aligned}$$

The product $Z_{\mu_{N-1}}(\beta_{N-1} + i\lambda)Z_{\mu_N}(\beta_N)$ has a simple pole at $\beta_N = \beta_{N-1} + i\lambda - \pi i$. Hence if one takes the limit $\lambda \rightarrow 2\pi$, the residue at the pole $\beta_N = \beta_{N-1} + \pi i$ is given by

$$\begin{aligned} & 2\pi i \text{ res } F^\mathcal{O}(\beta_1, \dots, \beta_N)_{\mu_1, \dots, \mu_N} \\ & = C F^\mathcal{O}(\beta_1, \dots, \beta_{N-2})_{\mu'_1, \dots, \mu'_{N-2}} \delta_{\mu_N + \mu_{N-1}, 0} \\ & \times \left(\delta_{\mu_1}^{\mu'_1} \cdots \delta_{\mu_{N-1}}^{\mu'_{N-1}} \right. \\ & \left. - S_{\mu_{N-1}\mu_{N-2}}^{\tau_1\mu'_{N-2}}(\beta_{N-1} - \beta_{N-2}) S_{\tau_1\mu_{N-3}}^{\tau_2\mu'_{N-3}}(\beta_{N-1} - \beta_{N-3}) \cdots S_{\tau_{N-2}\mu_1}^{\mu'_{N-1}\mu'_1}(\beta_{N-1} - \beta_1) \right). \end{aligned} \tag{5.7}$$

The other cases $\beta_N = \beta_j + \pi i$, $j \leq N-2$ follows (5.7) and the S -matrix symmetry. Note that the charge conjugation matrix $\mathcal{C}_{\mu, \mu'}$ has been set $\delta_{\mu+\mu', 0}$.

We hence conjecture that the function (5.5) provides a form factor of some local operator in the sine-Gordon theory. In addition, it admit the rational limit $\xi \rightarrow \infty$. In this limit, the function is expected to provide a form factor in the $SU(2)$ invariant Thirring model.

The boson realization discussed in §4 allows the evaluation of the trace. We here present only its final result. Setting $N = 2n$, $\lambda = 2\pi$, $\mu_j = -$ for $j = 1, 2, \dots, n$ and $\mu_j = +$ for $j = n+1, \dots, 2n$, we get

$$\begin{aligned}
& F^O(\beta_1, \dots, \beta_{2n})_{-, \dots, -, +, \dots, +} \\
&= \prod_{1 \leq r < s \leq 2n} \zeta(\beta_r - \beta_s) \prod_{j=1}^m \prod_{k=1}^{2n} \frac{1}{\sinh\left(\frac{\pi i}{4} - \frac{\beta_k - \alpha_j}{2}\right)} \\
&\times \prod_{l=1}^n \left(\int_{C_{\delta_l}} \frac{d\delta_l}{2\pi} \right) \mathcal{F}_{I-II}(\alpha_1, \dots, \alpha_m; \delta_1, \dots, \delta_n) \\
&\times \prod_{1 \leq l < j \leq 2n} \varphi(\delta_l - \beta_j + \pi i) \prod_{1 \leq j < l \leq n} \varphi(\beta_j - \delta_l + \pi i) \prod_{l=1}^n \frac{\varphi(\beta_l - \delta_l + \pi i)}{\sinh \frac{1}{\xi} \left(\beta_l - \delta_l - \frac{\pi i}{2} \right)} \\
&\times \prod_{1 \leq k < l \leq n} \sinh \frac{1}{\xi} (\delta_l - \delta_k - \pi i) \sinh(\delta_l - \delta_k). \tag{5.8}
\end{aligned}$$

Here

$$\zeta(\beta) = \sinh \frac{\beta}{2} \exp \left(\int_0^\infty \frac{dt \sin^2 \left((\beta + \pi i) \frac{t}{2} \right) \sinh(1 - \xi) \frac{\pi t}{2}}{t \sinh \frac{\pi t \xi}{2} \sinh \pi t \cosh \frac{\pi t}{2}} \right), \tag{5.9}$$

$$\varphi(\beta) = \exp \left(-2 \int_0^\infty \frac{dt \sin^2 \frac{\beta t}{2} \sinh(1 + \xi) \frac{\pi t}{2}}{t \sinh \frac{\pi t \xi}{2} \sinh \pi t} \right), \tag{5.10}$$

$$\begin{aligned}
& \mathcal{F}_{I-II}(\alpha_1, \dots, \alpha_m; \delta_1, \dots, \delta_n) \\
&= \prod_{a \in A} \left(\int_{C_{\gamma_a}} \frac{d\gamma_a}{2\pi} \prod_{j=1}^m \frac{1}{\cosh(\gamma_a - \alpha_j)} \right) \prod_{a' \in A'} \left(\int_{C_{\gamma'_{a'}}} \frac{d\gamma'_{a'}}{2\pi} \prod_{j=1}^m \frac{1}{\cosh(\gamma'_{a'} - \alpha_j)} \right) \\
&\times \prod_{a < b} \frac{\sinh(\gamma_a - \gamma_b)}{\sinh \nu(\gamma_a - \gamma_b - \pi i)} \prod_{a, a'} \frac{\sinh(\gamma'_{a'} - \gamma_a)}{\sinh \nu(\gamma'_{a'} - \gamma_a - \pi i)} \prod_{a' < b'} \frac{\sinh(\gamma'_{a'} - \gamma'_{b'})}{\sinh \nu(\gamma'_{a'} - \gamma'_{b'} - \pi i)} \\
&\times \prod_{a \in A} \left(\prod_{a < j} \sinh \nu(\gamma_a - \alpha_j + \frac{\pi i}{2}) \prod_{j < a} \sinh \nu(\alpha_j - \gamma_a + \frac{\pi i}{2}) \right) \\
&\times \prod_{a' \in A'} \left(\prod_{j < a'} \sinh \nu(\gamma'_{a'} - \alpha_j - \frac{\pi i}{2}) \prod_{a' < j} \sinh \nu(\alpha_j - \gamma'_{a'} + \frac{3\pi i}{2}) \right) \\
&\times \frac{\prod_{k=1}^{2n} \left(\prod_{a \in A} \sinh \frac{1}{2}(\beta_k - \gamma_a) \prod_{a' \in A'} \sinh \frac{1}{2}(\beta_k - \gamma'_{a'}) \right) \prod_{j=1}^m \prod_{l=1}^n \sinh(\delta_l - \alpha_j)}{\prod_{l=1}^n \left(\prod_{a \in A} \cosh(\delta_l - \gamma_a) \prod_{a' \in A'} \cosh(\delta_l - \gamma'_{a'}) \right)}. \tag{5.11}
\end{aligned}$$

In (5.11), $\nu = 1/(\xi + 1)$, $A = \{j | 1 \leq j \leq m, \varepsilon_j = -\}$ and $A' = \{j | 1 \leq j \leq m, \varepsilon_j = +\}$. The integration contours are determined from C_1 and C_2 in

(4.3) and (4.5) and the convergence region of the operator products of the vertex operators V, \bar{V}, V', \bar{V}' listed in Appendix of Ref.[1]. We chose them as follows.

The contour C_{δ_l} is $(-\infty, \infty)$ except that the poles at

$$\begin{aligned} &\beta_j - \frac{\pi i}{2} + 2\pi i n_1 + i\pi \xi n_2 \ (l < j), \ \beta_j + \frac{3\pi i}{2} + 2\pi i n_1 + i\pi \xi n_2 \ (j \leq l), \\ &\beta_l - \frac{\pi i}{2} + \pi i \xi n_2, \ \gamma_a - \frac{\pi i}{2} + i\pi n_1, \ \gamma'_{a'} - \frac{\pi i}{2} + i\pi n_1 \end{aligned}$$

$(n_1, n_2 \in \mathbf{Z}_{\geq 0})$ are above C_{δ_l} and the poles at

$$\begin{aligned} &\beta_j - \frac{3\pi i}{2} - 2\pi i n_1 - i\pi \xi n_2 \ (l < j), \ \beta_j + \frac{\pi i}{2} - 2\pi i n_1 - i\pi \xi n_2 \ (j \leq l), \\ &\gamma_a - \frac{3\pi i}{2} - i\pi n_1, \ \gamma'_{a'} - \frac{3\pi i}{2} - i\pi n_1 \end{aligned}$$

$(n_1, n_2 \in \mathbf{Z}_{\geq 0})$ are below C_{δ_l} .

The contour C_{γ_a} is $(-\infty, \infty)$ except that the poles at

$$\begin{aligned} &\alpha_j + \frac{\pi i}{2} + i\pi n \ (a \leq j), \ \alpha_j + \frac{3\pi i}{2} + i\pi n \ (j < a), \\ &\gamma_b + \pi i + (n+1)\frac{\pi i}{\nu} \ (a < b), \ \gamma_b - \pi i + (n+1)\frac{\pi i}{\nu} \ (b < a), \\ &\gamma_{a'} - \pi i + (n+1)\frac{\pi i}{\nu}, \ \delta_l + \frac{3\pi i}{2} + i\pi n \end{aligned}$$

$(n \in \mathbf{Z}_{\geq 0})$ are above C_{γ_a} and the poles at

$$\begin{aligned} &\alpha_j - \frac{3\pi i}{2} - i\pi n \ (a < j), \ \alpha_j - \frac{\pi i}{2} - i\pi n \ (j \leq a), \\ &\gamma_b + \pi i - (n+1)\frac{\pi i}{\nu} \ (a < b), \ \gamma_b - \pi i - (n+1)\frac{\pi i}{\nu} \ (b < a), \\ &\gamma_{a'} - \pi i - (n+1)\frac{\pi i}{\nu}, \ \delta_l + \frac{\pi i}{2} - i\pi n \end{aligned}$$

$(n \in \mathbf{Z}_{\geq 0})$ are below C_{γ_a} .

The contour $C_{\gamma'_{a'}}$ is $(-\infty, \infty)$ except that the poles at

$$\begin{aligned} &\alpha_j + \frac{5\pi i}{2} + i\pi n \ (a' < j), \ \alpha_j + \frac{3\pi i}{2} + i\pi n \ (j \leq a'), \\ &\gamma_a + \pi i + (n+1)\frac{\pi i}{\nu}, \ \gamma_{b'} + \pi i + (n+1)\frac{\pi i}{\nu} \ (a' < b'), \\ &\gamma_{b'} - \pi i + (n+1)\frac{\pi i}{\nu} \ (b' < a'), \ \delta_l + \frac{3\pi i}{2} + n\pi i \end{aligned}$$

$(n \in \mathbf{Z}_{\geq 0})$ are above $C_{\gamma_{a'}}$ and the poles at

$$\begin{aligned} & \alpha_j + \frac{\pi i}{2} - i\pi n \ (a' \leq j), \ \alpha_j - \frac{\pi i}{2} - i\pi n \ (j < a'), \\ & \gamma_a + \pi i - (n+1)\frac{\pi i}{\nu}, \ \gamma_{b'} + \pi i - (n+1)\frac{\pi i}{\nu} \ (a' < b'), \\ & \gamma_{b'} - \pi i - (n+1)\frac{\pi i}{\nu} \ (b' < a'), \ \delta_l + \frac{\pi i}{2} - n\pi i \end{aligned}$$

$(n \in \mathbf{Z}_{\geq 0})$ are below $C_{\gamma_{a'}}$.

In (5.8), we have omitted a constant factor, which depends on ξ . When one considers the rational limit, this factor should be properly considered. The rational limit yields the formula in §10.4 in Ref.[2] obtained as the rational limit of the form factor in the XXZ model in the antiferromagnetic regime and (7.14) in Ref.[18] with $\hbar = -i\pi$ as a special case $m = 1$.

The detail of the calculation and the relation with the Smirnov's integral formula will be discussed in elsewhere.

Acknowledgments

The author would like to thank the organizers of the workshop Yvan Saint-Aubin, Luc Vinet and Mo-Lin Ge for kind invitation and hospitality. He is also grateful to Michio Jimbo and Tetsuji Miwa for collaboration in the work [1] and for stimulating discussions.

6 Appendix

Here we list the formulas of the form

$$X(\beta_1)Y(\beta_2) = C_{X,Y}(\beta_2 - \beta_1) : X(\beta_1)Y(\beta_2) :$$

where $X, Y = V_R, \bar{V}_R, V'_R, \bar{V}'_R$ and $C_{X,Y}(\beta)$ is a meromorphic function on \mathbf{C} . The equality $C_{X,Y} = C_{Y,X}$ is valid in all cases. The whole expressions are obtained by taking the limit $\xi \rightarrow \infty$ in those listed in Ref.[1].

$$V_R(\beta_1)V_R(\beta_2) = g_R(\beta_2 - \beta_1) : e^{i\phi_R(\beta_1)+i\phi_R(\beta_2)} : \quad (\text{Im}(\beta_2 - \beta_1) < 0) \quad (6.1)$$

$$g_R(\beta) = \sqrt{2\pi}e^{\gamma/2} \frac{\Gamma(\frac{1}{2} + \frac{i\beta}{2\pi})}{\Gamma(\frac{i\beta}{2\pi})}$$

$$V_R(\beta_1)\bar{V}_R(\beta_2) = w_R(\beta_2 - \beta_1) : e^{i\phi_R(\beta_1)-i\bar{\phi}_R(\beta_2)} : \quad (\text{Im}(\beta_2 - \beta_1) < -\frac{\pi}{2}) \quad (6.2)$$

$$w_R(\beta) = \frac{e^{-\gamma}}{i(\beta + \frac{\pi i}{2})} = \frac{1}{g_R(\beta + \frac{\pi i}{2})g_R(\beta - \frac{\pi i}{2})}$$

$$\bar{V}_R(\beta_1)\bar{V}_R(\beta_2) = \bar{g}_R(\beta_2 - \beta_1) : e^{-i\bar{\phi}_R(\beta_1) - i\bar{\phi}_R(\beta_2)} : \quad (\text{Im}(\beta_2 - \beta_1) < -\pi) \quad (6.3)$$

$$\bar{g}_R(\beta) = -e^{2\gamma}\beta(\beta + \pi i) = \frac{1}{w_R(\beta + \frac{\pi i}{2})w_R(\beta - \frac{\pi i}{2})}$$

$$V'_R(\beta_1)V'_R(\beta_2) = g'_R(\beta_2 - \beta_1) : e^{i\phi'_R(\beta_1) + i\phi'_R(\beta_2)} : \quad (\text{Im}(\beta_2 - \beta_1) < \pi) \quad (6.4)$$

$$g'_R(\beta) = \sqrt{2\pi}e^{\gamma/2} \frac{\Gamma(1 + \frac{i\beta}{2\pi})}{\Gamma(\frac{1}{2} + \frac{i\beta}{2\pi})}$$

$$V'_R(\beta_1)\bar{V}'_R(\beta_2) = w'_R(\beta_2 - \beta_1) : e^{i\phi'_R(\beta_1) - i\bar{\phi}'_R(\beta_2)} : \quad (\text{Im}(\beta_2 - \beta_1) < \frac{\pi}{2}) \quad (6.5)$$

$$w'_R(\beta) = \frac{e^{-\gamma}}{i(\beta - \frac{\pi i}{2})} = \frac{1}{g'_R(\beta + \frac{\pi i}{2})g'_R(\beta - \frac{\pi i}{2})}$$

$$\bar{V}'_R(\beta_1)\bar{V}'_R(\beta_2) = \bar{g}'_R(\beta_2 - \beta_1) : e^{-i\bar{\phi}'_R(\beta_1) - i\bar{\phi}'_R(\beta_2)} : \quad (\text{Im}(\beta_2 - \beta_1) < 0) \quad (6.6)$$

$$\bar{g}'_R(\beta) = -e^{2\gamma}\beta(\beta - \pi i) = \frac{1}{w'_R(\beta + \frac{\pi i}{2})w'_R(\beta - \frac{\pi i}{2})}$$

$$V_R(\beta_1)V'_R(\beta_2) = h(\beta_2 - \beta_1) : e^{i\phi_R(\beta_1) + i\phi'_R(\beta_2)} : \quad (\text{Im}(\beta_2 - \beta_1) < \frac{\pi}{2}) \quad (6.7)$$

$$h(\beta) = \frac{\Gamma(\frac{i\beta}{2\pi} + \frac{1}{4})}{\Gamma(\frac{i\beta}{2\pi} + \frac{3}{4})} e^{-\frac{1}{2}(\gamma + \log(2\pi))}$$

$$V_R(\beta_1)\bar{V}'_R(\beta_2) = i(\beta_2 - \beta_1)e^\gamma : e^{i\phi_R(\beta_1) - i\bar{\phi}'_R(\beta_2)} : \quad (\text{Im}(\beta_2 - \beta_1) < 0) \quad (6.8)$$

$$\bar{V}_R(\beta_1)V'_R(\beta_2) = i(\beta_2 - \beta_1)e^\gamma : e^{-i\bar{\phi}_R(\beta_1) + i\phi'_R(\beta_2)} : \quad (\text{Im}(\beta_2 - \beta_1) < 0) \quad (6.9)$$

$$\bar{V}_R(\beta_1)\bar{V}'_R(\beta_2) = -\frac{e^{-2\gamma}}{(\beta_2 - \beta_1)^2 + \frac{\pi^2}{4}} : e^{-i\bar{\phi}_R(\beta_1) - i\bar{\phi}'_R(\beta_2)} : \quad (\text{Im}(\beta_2 - \beta_1) < -\frac{\pi}{2}) \quad (6.10)$$

We set

$$S_{R0}(\beta) = \frac{g_R(-\beta)}{g_R(\beta)}, \quad R_{R0}(\beta) = \frac{g'_R(-\beta)}{g'_R(\beta)}. \quad (6.11)$$

The following relations are valid.

$$\frac{w_R(\beta)}{w_R(-\beta)} = -\frac{\beta - \frac{\pi i}{2}}{\beta + \frac{\pi i}{2}}, \quad (6.12)$$

$$\frac{\bar{g}_R(\beta)}{\bar{g}_R(-\beta)} = \frac{\beta + \pi i}{\beta - \pi i}, \quad (6.13)$$

$$\frac{w'_R(\beta)}{w'_R(-\beta)} = -\frac{\beta + \frac{\pi i}{2}}{\beta - \frac{\pi i}{2}}, \quad (6.14)$$

$$\frac{\bar{g}'_R(\beta)}{\bar{g}'_R(-\beta)} = \frac{\beta - \pi i}{\beta + \pi i}, \quad (6.15)$$

$$\frac{h(\beta)}{h(-\beta)} = -\frac{\sinh(\frac{\beta}{2} - \frac{\pi i}{4})}{\sinh(\frac{\beta}{2} + \frac{\pi i}{4})}. \quad (6.16)$$

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